A SAMPLE RESEARCH PAPER/THESIS/DISSERTATION ON ASPECTS OF ELEMENTARY LINEARY ALGEBRA

by

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AN ABSTRACT OF THE DISSERTATION OF

NAME OF STUDENT, for the Doctor of Philosophy degree in MAJOR FIELD, presented on DATE OF DEFENSE, at Southern Illinois University Carbondale. (Do not use abbreviations.)

TITLE: A SAMPLE RESEARCH PAPER ON ASPECTS OF ELEMENTARY LINEAR ALGEBRA

MAJOR PROFESSOR: Dr. J. Jones

(Begin the abstract here, typewritten and double-spaced. A thesis abstract should consist of 350 words or less including the heading. A page and one-half is approximately 350 words.)

DEDICATION

(NO REQUIRED FOR RESEARCH PAPER)

(The dedication, as the name suggests is a personal dedication of one's work. The section is OPTIONAL and should be double-spaced if included in the thesis/dissertation.)

ACKNOWLEDGMENTS

(NOT REQUIRED IN RESEARCH PAPER)

I would like to thank Dr. Jones for his invaluable assistance and insights leading to the writing of this paper. My sincere thanks also goes to the seventeen members of my graduate committee for their patience and understanding during the nine years of effort that went into the production of this paper.

A special thanks also to Howard Anton [1], from whose book many of the examples used in this sample research paper have been quoted. Another special thanks to Prof. Ronald Grimmer who provided the previous thesis template upon which much of this is based and for help with graphics packages.

PREFACE

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A preface or foreword may contain the author's statement of the purpose of the study or special notes to the reader. This section is OPTIONAL and should be double-spaced if used in the thesis/dissertation.

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INTRODUCTION

This paper provides an elementary treatment of linear algebra that is suitable for students in their freshman or sophomore year. Calculus is not a prerequisite.

The aim in writing this paper is to present the fundamentals of linear algebra in the clearest possible way. Pedagogy is the main consideration. Formalism is secondary. Where possible, basic ideas are studied by means of computational examples and geometrical interpretation.

The treatment of proofs varies. Those proofs that are elementary and hve significant pedagogical content are presented precisely, in a style tailored for beginners. A few proofs that are more difficult, but pedgogically valuable, are placed at the end of of the section and marked "Optional". Still other proofs are omitted completely, with emphasis placed on applying the theorem.

Chapter 1 deals with systems of linear equations, how to solve them, and some of their properties. It also contains the basic material on matrices and their arithmetic properties.

Chapter 2 deals with determinants. I have used the classical permutation approach. This is less abstract than the approach through n-linear alternative forms

and gives the student a better intuitive grasp of the subject than does an inductive development.

Chapter 3 introduces vectors in 2-space and 3-space as arrows and develops the analytic geometry of lines and planes in 3-space. Depending on the background of the students, this chapter can be omitted without a loss of continuity.

CHAPTER 1

SYSTEMS OF LINEAR EQUATIONS AND MATRICES

1.1 INTRODUCTIONS TO SYSTEMS OF LINEAR EQUATIONS

In this section we introduce base terminology and discuss a method for solving systems of linear equations.

A line in the xy-plain can be represented algebraically by an equation of the form

$$a_1x + a_2y = b$$

An equation of this kind is called a linear equation in the variables x and y. More generally, we define a linear equation in the n variables $x_1, \dots, x_n$ to be one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1.1}$$

where $a_1, a_2, \dots a_n$ and b are real constants.

Definition. A finite set of linear equations in the variables $x_1, x_2, \dots, x_n$ is called a *system of linear equations.*

Not all systems of linear equations has solutions. A system of equations that has no solution is said to be *inconsistent.* If there is at least one solution, it is called *consistent.* To illustrate the possibilities that can occur in solving systems of linear equations, consider a general system of two linear equations in the unknowns x and y:

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

The graphs of these equations are lines; call them l_1 and l_2. Since a point (x, y) lies on a line if and only if the numbers x and y satisfy the equation of the line, the solutions of the system of equations will correspond to points of intersection of l_1 and l_2. There are three possibilities:

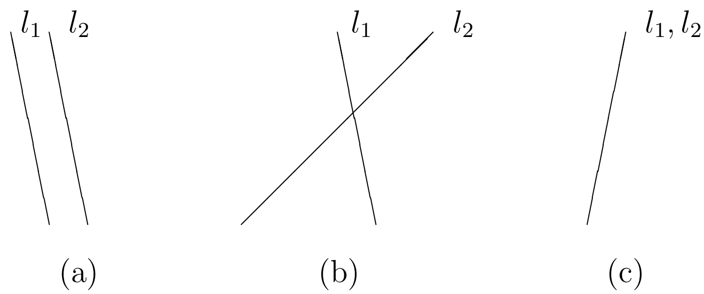


Figure 1.1. (a) no solution, (b) one solution, (c) infinitely many solutions

The three possiblities illustrated in Figure 1.1 are as follows:

(a) l_1 and l_2 are parallel, in which case there is no intersection, and consequently no solution to the system.

(b) l_1 and l_2 intersect at only one point, in which case the system has exactly one solution.

(c) l_1 and l_2 coincide, in which case there are infinitely many points of intersection, and consequently infinitely many solutions to the system.

Although we have considered only two equations with two unknowns here, we will show later that this same result holds for arbitrary systems; that is, every system of linear equations has either no solutions, exactly one solution, or infinitely many solutions.

1.2 GAUSSIAN ELIMINATION

In this section we give a systematic procedure for solving systems of linear equations; it is based on the idea of reducing the augmented matrix to a form that is simple enough so that the system of equations can be solved by inspection.

Remark. It is not difficult to see that a matrix in row-echelon form must have zeros below each leading 1. In contrast a matrix in reduced row-echelon form must have zeros above and below each leading 1.

As a direct result of Figure 1.1 on page 3 we have the following important theorem.

Theorem 1.2.1. *A homogenous system of linear equations with more unknowns than equations always has infinitely many solutions*

The definition of matrix multiplication requires that the number of columns of the first factor A be the same as the number of rows of the second factor B in order to form the product AB. If this condition is not satisfied, the product is undefined. A convenient way to determine whether a product of two matrices is defined is to write down the size of the first factor and, to the right of it, write down the size of the second factor. If, as in Figure 1.2, the inside numbers are the same, then the product is defined. The outside numbers then give the size of the product.

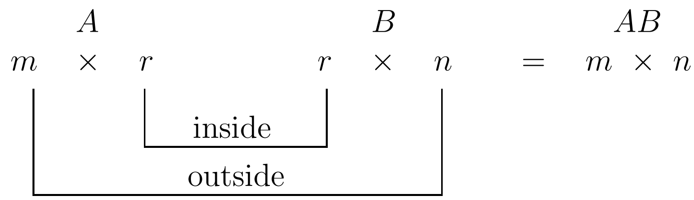


Figure 1.2. Inside and outside numbers of a matrix multiplication problem of $A \times B = AB$, showing how the inside dimensions are dropped and the dimensions of the product are the outside dimenions.

Although the commutative law for multiplication is not valid in matrix arithmetic, many familiar laws of arithmetic are valid for matrices. Some of the most important ones and their names are summarized in the following proposition.

Proposition 1.2.2. *Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.*

(a) $A + B = B + A$ *(Commutative law for addition)*

(b) $A + (B + C) = (A + B) + C$ *(Associative law for addition)*

(c) $A(BC) = (AB)C$ *(Associative law for multiplication)*

1.3 FURTHER RESULTS ON SYSTEMS OF EQUATIONS

In this section we shall establish more results about systems of linear equations and invertibility of matrices. Our work will lead to a method for solving n equations in n unknowns that is more efficient than Gaussian elimination for certain kinds of problems.

1.3.1 Some Important Theorems

Theorem 1.3.1. *If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix B, the system of equations $AX = B$ has exactly one solution, namely, $X = A^{-1}B$.*

Proof. Since $A(A^{-1}B) = B$, $X = A^{-1}B$ is a solution of $AX = B$. To show that this is the only solution, we will assume that X_0 is an arbitrary solution, and then show that X_0 must be the solution $A^{-1}B$.

If X_0 is any solution, then $AX_0 = B$. Multiplying both sides by A^{-1}, we obtain $X_0 = A^{-1}B$. □

Theorem 1.3.2. *Let A be a square matrix.*

(a) If B is a square matrix satisfying $BA = I$,then $B = A^{-1}$.

(b) If B is a square matrix satisfying $AB = I$, then $B = A^{-1}$.

In our later work the following fundamental problem will occur over and over again in various contexts.

Let A be fixed $m \times n$ matrix. Find all $m \times 1$ matrices B such that the system of equations $AX = B$ is consistent.

If A is an invertible matrix, Theorem 1.3.2 completely solves this problem by asserting that for every $m \times n$ matrix B, $AX = B$ has the unique solution $X = A^{-1}B$.

CHAPTER 2

DETERMINANTS

2.1 THE DETERMINANT FUNCTION

We are all familiar with functions lilke $f(x) = \sin x$ and $f(x) = x^2$, which associate a real number $f(x)$ with a real value of the variable x. Since both x and $f(x)$ assume only real values, such functions can be described as real-valued functions of a matrix variable, that is, functions that associate a real number $f(X)$ with a matrix X.

Before we shall be able to define the determinant function, it will be necessary to establish some results concerning permutations.

Definition. A *permutation* of the set of integers $\{1, 2, \ldots, n\}$ is an arrangement of these integers in some order without ommissions or repetitions.

Example 2.1.1. There are six different permutations of the set of integers $\{1, 2, 3\}$. These are

$$(1,2,3)(2,1,3)(3,1,2) \tag{2.1}$$

$$(1,3,2)(2,3,1)(3,2,1) \tag{2.2}$$

One convenient method of systematically listing permutations is to use a *permutation tree*. This method will be illustrated in our next example.

Example 2.1.2. List all permutations of the set of integers $\{1, 2, 3, 4\}$

Solution. By drawing a permutation tree with each branch representing all four numbers, we see that there are a total of 24 possible permutations.

2.2 EVALUATING DETERMINANTS BY ROW REDUCTION

In this section we show that the determinant of a matrix can be evaluated by reducing the matrix to row-echelon form. This method is of importance since it avoids the lengthy computations involved directly applying the determinant definition.

We first consider two class of matrices whose determinants can be easily evaluated, regardless of the size of the matrix.

Theorem 2.2.1. *If A is any square matrix that contains a row of zeros, then* $\det(A) = 0$.

Theorem 2.2.2. *If A is an $n \times n$ triangular matrix, then* $\det(A)$ *is the product of the entries on the main diagonal; that is* $\det(A) = a_{11}a_{22}\cdots a_{mn}$.

Example 2.2.1. Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \tag{2.3}$$

2.2.1 Some Final Conclusions

It should be evident from the examples in this section that whenever a square matrix has two proportional rows (like the first and second rows of A), it is possible to introduce a row of zeros by adding a suitable mutliple of one of these rows to the other. Thus, if a square matrix has two proportional rows, its determinant is zero.

In the next section we consider some examples of linear algebra functions expressed in table form – primarily to see the list of tables command works in Latex.

2.3 PROPERTIES OF THE DETERMINANT FUNCTION

In this section we develop some of the fundamental properties of the determinant function. our work here will give us some further insight into the relationship between a square matrix and its determinant. One of the immediate consequences of this material will be an important determinant test for the invertibility of a matrix

Consider Table 2.1 and its implications in the area of linear algebra.

Function	1	2	3
Value	2.45	34.12	1.00
Determinant	0	0	0
Inverse	1	1	1

Table 2.1. An example table showing how centering works with extended captioning.

It should be evident from the examples in this section that whenever a square matrix has two proportional rows (like the first and second rows of A), it is possible to introduce a row of zeros by adding a suitable mutliple of one of these rows to the other. Thus, if a square matrix has two proportional rows, its determinant is zero.

We hope this has given some insights into the basics of linear algebra and its impact on the world around us. We leave you now with two encapsulated postscript graphs which illustrate the main points discussed in this paper.

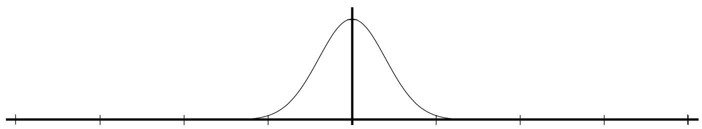


Figure 2.1. An encapsulated postscript file.

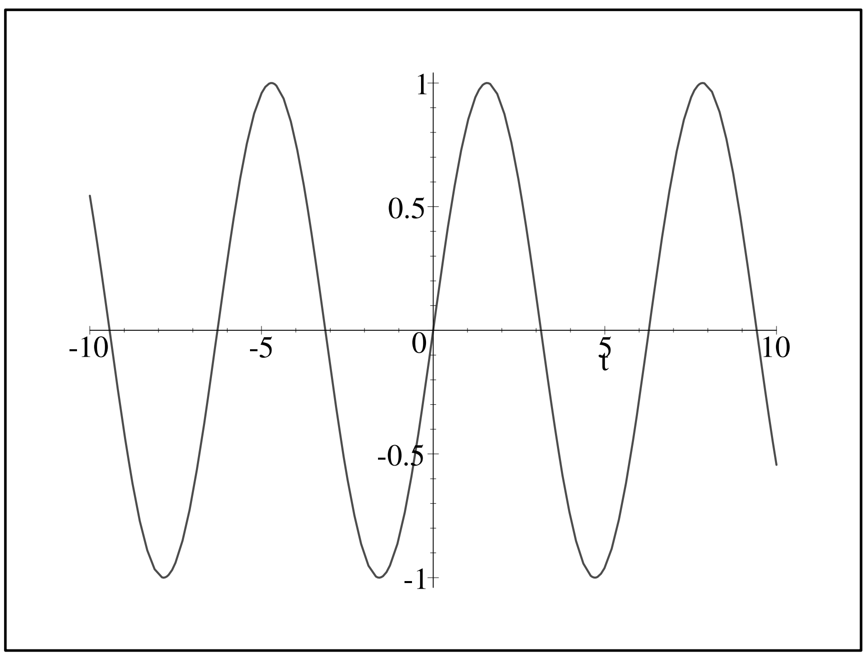


Figure 2.2. A second encapsulated postscript file.

CHAPTER 3

EXAMPLES

Some examples of the definitions found in the file ps-defs.tex follow below.

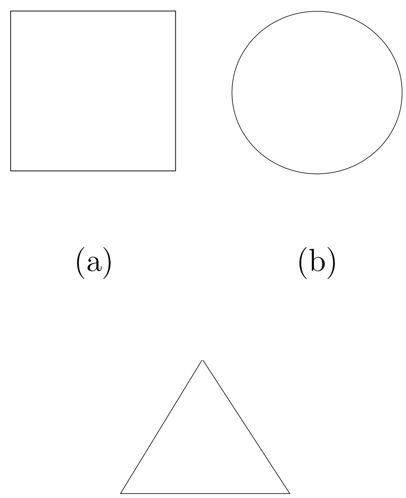
Here are examples of how you can use equation numbers with multiple line equations.

$$\begin{aligned}
(f+(g+h))(a) &= f(a)+(g+h)(a) \\
&= f(a)+(g(a)+h(a)) \qquad (3.1)\\
&= (f(a)+g(a))+h(a) \\
&= (f+g)(a)+h(a) \\
&= ((f+g)+h)(a) \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
(f+(g+h))(a) &= f(a)+(g+h)(a) \\
&= f(a)+(g(a)+h(a)) \\
&= (f(a)+g(a))+h(a) \qquad (3.3)\\
&= (f+g)(a)+h(a) \\
&= ((f+g)+h)(a)
\end{aligned}$$

$$\begin{aligned}
(f+(g+h))(a) &= f(a)+(g+h)(a) \\
&= f(a)+(g(a)+h(a)) \\
&= (f(a)+g(a))+h(a) \\
&= (f+g)(a)+h(a) \\
&= ((f+g)+h)(a)
\end{aligned}$$

Below is a figure which shows how to line up small figures on multiple lines. The .dvi version is immediately below. The .pdf version may be found underneath the complete figure and commented out. If you exchange the sections commented out, then you can compile a .pdf file.



(a) (b)

(c)

Figure 3.1. Two rows of graphics: (a) Square (b) Circle (c) Rectangle

Three figures across the page requires fairly small figures to fit within the Graduate School margins.

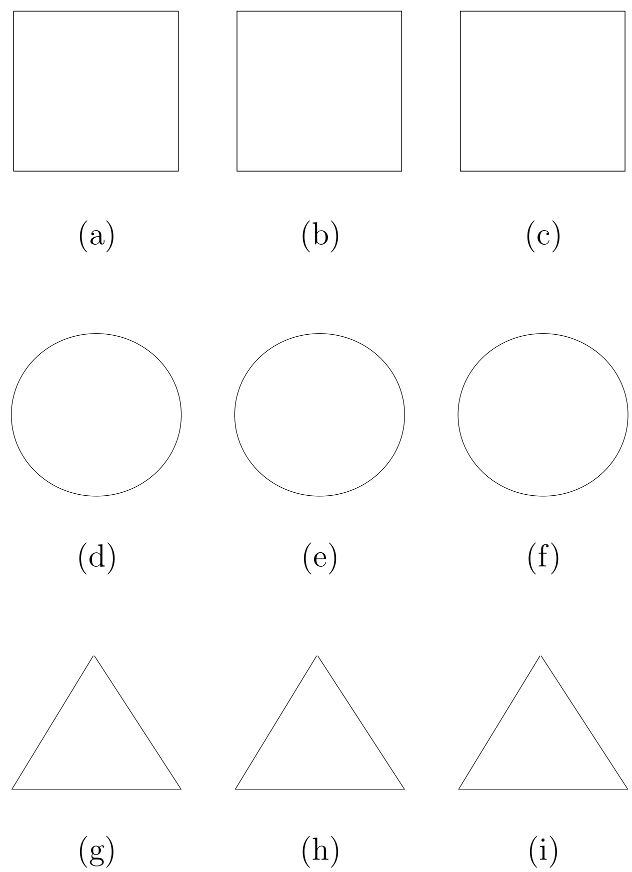


Figure 3.2. Three rows of graphics: (a)–(c) Squares. (d)–(f) Circles. (g)–(i) Ovals.

The verbatim environment can be useful when using data from a spreadsheet as is done below.

```
 X,TRUE_SUR,MSE SIM,MSE ZHAO,MSE JIAN,MSE PZHAO,MSE PJIAN
 0.0  0.7520  0.03864  0.01407  0.01407  0.01180  0.01223
 4.0  0.7273  0.04079  0.01675  0.01675  0.01479  0.01551
 8.0  0.7035  0.04203  0.01923  0.01923  0.01675  0.01817
12.0  0.6524  0.04581  0.02157  0.02135  0.01932  0.02043
16.0  0.6029  0.05146  0.02345  0.02266  0.02304  0.02320
20.0  0.5551  0.05343  0.02498  0.02393  0.02627  0.02509
24.0  0.5089  0.05449  0.02677  0.02453  0.02936  0.02641
28.0  0.4641  0.05706  0.02901  0.02442  0.03315  0.02722
32.0  0.4209  0.05719  0.02910  0.02341  0.03558  0.02776
36.0  0.3790  0.05656  0.02974  0.02229  0.03745  0.02667
40.0  0.3385  0.05518  0.02940  0.02119  0.03864  0.02618
44.0  0.2994  0.05344  0.02989  0.02054  0.03928  0.02531
48.0  0.2615  0.04950  0.02803  0.01906  0.03855  0.02414
52.0  0.2249  0.04582  0.02712  0.01812  0.03849  0.02229
56.0  0.1895  0.04101  0.02454  0.01578  0.03632  0.01918
60.0  0.1552  0.03564  0.02282  0.01315  0.03372  0.01629
64.0  0.1220  0.03216  0.02124  0.00997  0.03188  0.01391
68.0  0.0900  0.02420  0.01730  0.00688  0.02551  0.01070
72.0  0.0590  0.01592  0.01254  0.00363  0.01811  0.00622
76.0  0.0290  0.00865  0.00838  0.00110  0.00886  0.00368
```

Figure 3.3. Use of verbatim environment

On the following page is an example of how to rotate text that is too long to fit within the horizontal margins that are required.

$$A = \begin{pmatrix} -(\hat{\theta}_{D_1}(3;1) - \hat{\theta}_{D_1}(1;1)) & 0 & 0 & \cdots & 0 & 0 \\ (\hat{\theta}_{D_1}(3;1) - \hat{\theta}_{D_1}(1;1)) & -(\hat{\theta}_{D_2}(3;1) - \hat{\theta}_{D_2}(1;1)) & 0 & \cdots & \cdots & 0 \\ 0 & (\hat{\theta}_{D_2}(3;1) - \hat{\theta}_{D_2}(1;1)) & -(\hat{\theta}_{D_3}(3;1) - \hat{\theta}_{D_3}(1;1)) & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & (\hat{\theta}_{D_{n-1}}(3;1) - \hat{\theta}_{D_{n-1}}(1;1)) & -(\hat{\theta}_{D_n}(3;1) - \hat{\theta}_{D_n}(1;1)) \\ 0 & 0 & 0 & 0 & 0 & (\hat{\theta}_{D_n}(3;1) - \hat{\theta}_{D_n}(1;1)) \end{pmatrix},$$

$$\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix},$$

Figure 3.4. Matrix Rotated 90 degrees.

REFERENCES

[1] Anton, H., *Elementary Linear Algebra*, John Wiley & Sons, New York, 1977.

[2] Huang, X. and Krantz, S.G., *On a problem of Moser*, Duke Math. J. **78** (1995), 213–228.

[3] Kellog, O.D., *Harmonic functions and Green's integral*, Trans. Amer. Math. Soc. **13** (1912), 109–132.

[4] Kruzhilin, N.G., *Local automorphisms and mappings of smooth strictly pseudo-convex hypersurfaces*, (Russian), Dokl. Akad. Nauk SSSR **271** (1983), 280–282.

[5] Macdonald, I.G., *Symmetric Functions and Hall Polynomials*, Second edition, Clarendon, Oxford, 1995.

[6] Rasulov, K.M., Dokl. Akad. Nauk SSSR **252** (1980), 1059–1063; English transl. in Soviet Math. Dokl. **21** (1980).

[7] Rasulov, K.M., Dokl. Akad. Nauk SSSR **270** (1983), 1061–1065; English transl. in Soviet Math. Dokl. **27** (1983).

APPENDICES

APPENDIX I

APPENDIX II

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Major Professor: Dr. J. Jones

Publications: (OMIT IF NONE)